CONSTRUCTION OF EXACT DISCONTINUOUS SOLUTIONS OF THE EQUATIONS OF ONE-DIMENSIONAL GAS DYNAMICS AND THEIR APPLICATIONS

(POSTROENIYE TOCHNYKH RAZRYVNYKH URAVNENII ODNOMERNOI GAZODINAMIKI I IKH PRIMENENIIA)

PMM Vol.22, No.2, 1958, pp.265-268

V.P. KOROBEINIKOV and E.V. RIASANOV
(Moscow)

(Received 22 October 1957)

In the study of the properties of solutions of the equations of one-dimensional unsteady motion of a perfect gas in the presence of shock waves, discontinuous exact solutions are of great interest.

At the present time, exact discontinuous solutions are obtained only in special cases of self-similar problems [1]. To obtain new exact solutions, the particular solution of the equations of gas dynamics published by Sedov [1, 2] may be used, namely

$$v = -\frac{1}{\mu}\frac{d\mu}{dt}r, \qquad p = \mu^{\gamma\nu}\left\{C + \frac{\nu(\gamma-1)}{2(s+2)}BP(x)\right\}$$
$$\rho = \mu^{\nu}\xi^{s}P'(x), \qquad \frac{d\mu}{dt} = \pm\mu^{2}\left(A + B\mu^{\nu(\gamma-1)}\right)^{1/2} \qquad (1)$$

Here v is the velocity, ρ the density, p the pressure, $P(x)$ an arbitrary function, r the distance from the center of symmetry, t the time, $\mu = \mu(t)$ a function of time, A, B, C are arbitrary constants, s is a constant, $\nu = 1, 2, 3$ corresponds to the case of plane, cylindrical and spherical waves, respectively, γ is the adiabatic index, $\xi = r\mu$ is the Lagrangian coordinate, $x = \xi^{s+2}$.

An attempt to employ the Sedov solution for the construction of solutions with shock waves was made by Keller [3]. Below a method of solution is developed for the case when the shock wave is propagated through a gas at rest, whose density $\rho_1 = \rho_1(r)$ is variable and whose pressure p_1 is constant. If $r_2(t)$ is the radius of the shock wave, then let

$$v_2 = v(t, r_2), \qquad \rho_2 = \rho(t, r_2), \qquad p_2 = p(t, r_2)$$

To construct a closed solution, it is necessary to determine the law of motion of the shock wave $r_2(t)$ and to find the function $P(x)$.

We shall assume further that the function $\rho_1(r)$ is known in advance. The unknown functions $r_2(t)$, $P(x)$, $\rho_1(r)$ will be determined from the

Reprint Order No. PMM 32.

requirement that the solution (1) satisfies the boundary conditions at the front of the shock wave

$$v_2 = \frac{2}{\gamma+1}(1-q)c, \qquad \rho_2 = \frac{\gamma+1}{\gamma-1+2q}\rho_1, \qquad p_2 = \frac{p_1}{\gamma+1}\,\frac{2\gamma-(\gamma-1)q}{q} \tag{2}$$

where

$$c = \frac{dr_2}{dt}, \qquad q = \frac{\gamma p_1}{\rho_1 c^2}$$

From the first equation (1) and the first condition (2) we have

$$q = 1 + \frac{\gamma+1}{2}\,\frac{r_2}{\mu}\,\frac{d\mu}{dr_2} \tag{3}$$

Using the second and the third condition (2) and the values of ρ_2 and p_2 from (1), we may eliminate the arbitrary function $P(x)$. We obtain then the equation

$$q' = -q\left\{\frac{\nu}{2\mu}\left[2\gamma-(\gamma-1)q\right] + \frac{B\nu(\gamma-1)(\gamma+1)^2}{8(\gamma-1+2q)}\,\frac{(r_2{}^2\mu^2)'\,\mu^{\nu(\gamma-1)-4}}{(r_2')^2\left[A+B\mu^{\nu(\gamma-1)}\right]}\right\} \tag{4}$$

This procedure to eliminate the arbitrary function $P(x)$ was indicated to the authors by Sedov.

Primes in equation (4) indicate differentiation with respect to μ. In the following μ will be considered as the independent variable.

Eliminating the function $q(\mu)$ from (3) and (4), and introducing the substitution $y = (\ln r_2)'$, we obtain a first order Riccati equation for $y(\mu)$

$$\frac{dy}{d\mu} = \nu y^2 + \frac{1}{\mu}\left[\nu - 1 + \frac{\nu(\gamma-1)}{2}\,\frac{\mu^{\nu(\gamma-1)}}{\varkappa+\mu^{\nu(\gamma-1)}}\right]y - \frac{\varkappa(\gamma^2-1)\nu}{4\mu^2\left[\varkappa+\mu^{\nu(\gamma-1)}\right]}, \qquad \varkappa = \frac{A}{B} \tag{5}$$

Knowing the solution $y = y(\mu)$ of this equation we may, using formula (3), find the function $q(\mu)$ or $q(r_2)$, and therefore, also $\rho_1(r)$.

Having determined $p_2(\xi_2)$ and $\rho_2(\xi_2)$ by formulas (2), it becomes possible, using (1), to find the function $P(x)$, that is, to solve completely the stated problem. The solution of equation (5) for $\kappa \neq 0$ and arbitrary γ is not expressible in simple form through elementary functions.

Let us consider several special cases.

1). $\kappa = 0$. In this case the value of the quantity B is immaterial and it can be taken equal to unity.

Equation (5) is easily integrated and has the solution

$$y(\mu) = \mu^{\frac{1}{2}\nu(\gamma+1)-1}\left\{c_1\left[1 - \frac{2}{\gamma+1}\,\frac{1}{c_1}\,\mu^{\frac{1}{2}\nu(\gamma+1)}\right]\right\}^{-1} \tag{6}$$

From this the functions $r_2(\mu)$ and $q(\mu)$ are easily found

$$r_2(\mu)=c_2\left[1-\frac{2}{\gamma+1}\frac{1}{c_1}\mu^{\frac{\nu}{2}(\gamma+1)}\right]^{-\frac{1}{\nu}},\qquad q(\mu)=\frac{\gamma+1}{2}c_1\mu^{-\frac{\nu}{2}(\gamma+1)} \tag{7}$$

Here c_1 and c_2 are the constants of integration.

From formula $\rho_1 = \gamma p_1/c^2 q$ we can find $\rho_1(\mu)$. Eliminating μ from the functions $r_2(\mu)$ and $\rho_1(\mu)$ we obtain

$$\rho_1(r_2)=\gamma p_1 c_2^{2\nu}\left[\left(\frac{\gamma+1}{2}\right)^{\beta+1}c_1^{\beta-1}r_2^{\omega}(r_2^{\nu}-c_2^{\nu})^{\beta}\right]^{-1} \tag{8}$$

where

$$\beta=\frac{3\gamma\nu+4-\nu}{\nu(\gamma+1)},\qquad \omega=\frac{\nu(3-\gamma)+2(\gamma-1)}{\gamma+1}$$

The function $\mu(t)$ in this case is of the form

$$\mu(t)=[c_3\mp kt]^{-\frac{1}{k}},\qquad k=\tfrac{1}{2}\nu(\gamma-1)+1 \tag{9}$$

where c_3 is a constant of integration. Using (7) and (9) we find the law of motion of the shock wave

$$r_2(t)=c_2\left[1-\frac{2}{\gamma+1}\frac{1}{c_1}(c_3\mp kt)^{-\frac{\nu(\gamma+1)}{2k}}\right]^{-\frac{1}{\nu}} \tag{10}$$

Using formulas (1), (2) and (7) it is a simple matter to determine all the characteristics of motion at the front of the shock wave

$$p_2=p_1\left[1-\frac{2\gamma}{\gamma+1}\left(\frac{c_2}{r_2}\right)^{\nu}\right]$$

$$v_2=\mp r_2\left\{(\gamma+1)c_1\left[1-\left(\frac{c_2}{r_2}\right)^{\nu}\right]\right\}^{\chi}\qquad\left(\chi=\frac{\nu(\gamma-1)+2}{\nu(\gamma+1)}\right) \tag{11}$$

$$\rho_2=\frac{2\gamma p_1 c_2^{2\nu}}{(\gamma+1)r_2^{2(\nu+1)}}\left[\frac{\gamma+1}{2}-\frac{\gamma-1}{2}\left(\frac{c_2}{r_2}\right)^{\nu}\right]^{-1}\left\{\left(1-\left(\frac{c_2}{r_2}\right)^{\nu}\right)\frac{c_1(\gamma+1)}{2}\right\}^{-\chi}$$

Let us now find the arbitrary function $P(x)$. Since $\xi_2 = r_2\mu$ we obtain from (7)

$$c_2^{\nu}\varphi+\frac{2}{\gamma+1}\frac{1}{c_1}\varphi^{\frac{\gamma+1}{2}}x_2^{\frac{\nu}{s+2}}-x_2^{\frac{\nu}{s+2}}=0\qquad(\varphi(x)=\mu^{\nu}(x))$$

From equations (1), (2), (7) we obtain

$$P(x_2)=\frac{2(s+2)}{\nu(\gamma-1)}\left[\frac{p_1}{\gamma+1}\left(\frac{1-\gamma}{\mu^{\nu\gamma}}+\frac{4\gamma}{\gamma+1}\frac{1}{c_1}\mu^{\frac{\nu}{2}(1-\gamma)}\right)-C\right]$$

Thus, to satisfy the boundary conditions (2), $P(x)$ has to be taken in the form

$$P(x)=\frac{2(s+2)}{\nu(\gamma-1)}\left[\frac{p_1}{\gamma+1}\left(\frac{1-\gamma}{\varphi^{\gamma}}+\frac{4\gamma}{\gamma+1}\frac{1}{c_1}\varphi^{\frac{1-\gamma}{2}}\right)-C\right] \tag{12}$$

where $\phi(x)$ is to be found from the equation

$$c_2^{\nu}\varphi + \frac{2}{\gamma+1}\frac{1}{c_1}\varphi^{\frac{\gamma+1}{2}} x^{\frac{\nu}{s+2}} - x^{\frac{\nu}{s+2}} = 0 \tag{13}$$

2). $B = 0$. In this case we find from (4)

$$q(\mu) = \frac{2\gamma}{\gamma-1}\frac{1}{1+c_1\mu^{\gamma\nu}} \qquad \left(c_1 = \frac{C}{p_1}\frac{\gamma+1}{\gamma-1}\right)$$

From (1) and (3) we obtain

$$r_2(t) = \frac{1}{c_2} A^{\frac{\gamma+1}{4}} (t+t_0)^{\frac{\gamma+1}{2}} \left[1 + k_2 A^{\frac{\gamma\nu}{2}} (t+t_0)^{\gamma\nu}\right]^{-\frac{1}{\nu}}$$

Just as in the previous case, it is easy to find $\rho_1(r_2)$, $v_2(r_2)$, $p_2(r_2)$, $\rho_2(r_2)$, as well as the form of the arbitrary function $P'(x)$.

3). $\gamma = 1$. Equation (4) can be integrated in this case. A study of this solution will not be presented here. The general solution of equation (5) for $\kappa \neq 0$ and arbitrary γ may be obtained, using some particular solution.

We now proceed to the evaluation of the energy. The law of conservation of energy may be written down in the form

$$E + \frac{\sigma_\nu p_1}{\nu(\gamma-1)}(r''^{\nu} - r'^{\nu}) = \sigma_\nu \int_{r'}^{r''} \left(\frac{\rho v^2}{2} + \frac{p}{\gamma-1}\right) r^{\nu-1} dr \tag{14}$$

where E is the energy evolved in a certain period of time in a volume enclosed by radii r' and r'', and different from kinetic or thermal energies of the gas (this could be, for example, the energy given off in an explosion)

$$\sigma_\nu = 2\pi(\nu-1) + (\nu-2)(\nu-3)$$

The second term in the left-hand side of equation (14) determines the initial internal energy of the gas.

The right-hand side of equation (14) represents the energy of the gas, which was set in motion by the shock wave.

Using (1) and transforming the integral on the right-hand side of (14), we obtain a simple expression for the calculation of the energy balance

$$\frac{E}{\sigma_\nu} = \frac{p_1}{\nu(\gamma-1)}(r'^{\nu} - r''^{\nu}) + \frac{p(r'',t)\, r''^{\nu} - p(r',t)\, r'^{\nu}}{\nu(\gamma-1)} + \\ + \frac{A\mu^{\nu}}{2(s+2)}(r^{\nu}P)\Big|_{r'}^{r''} - \frac{A\nu\mu^{\nu}}{2(s+2)}\int_{r'}^{r''} P r^{\nu-1} dr \tag{15}$$

Employing the results obtained above, it is possible to solve a non-self-similar problem of a point-blast in a gas, whose initial density is variable.

In fact, from (1) and (15), letting $A = 0$, $r' = 0$, $r'' = r_2$ and assuming that E is the energy given off instantaneously in a blast, we obtain

$$p_2 = p_1\left[1 + \frac{\nu(\gamma-1)}{\sigma_\nu}\frac{E}{p_1}\frac{1}{r_2{}^\nu}\right] \tag{16}$$

From (8), (11) and (16) we find the initial density distribution

$$\rho_1(r) = \frac{b(\gamma-1)^2}{\gamma r^\omega}\left(\frac{\gamma+1}{2}\right)^{1-\beta}\left(r^\nu + \frac{r^{0\nu}(\gamma^2-1)\nu}{2\sigma_\nu\gamma}\right)^{-\beta}$$

$$b = \frac{\nu^2 r^{02\nu} p_1}{\sigma_\nu{}^2 c_1{}^{\beta-1}}, \qquad r^0 = \left(\frac{E}{p_1}\right)^{\frac{1}{\nu}} \tag{17}$$

where r^0 is the dynamical length.

From (17) it is seen that $\rho_1(r)$ depends parametrically on γ and r^0. Noting that $r_2(0) = 0$, we obtain $c_3 = 0$. Taking $v > 0$ and using (1), (12), (13), we find that the solution of this problem is of the form

$$v = \frac{r}{kt}, \qquad p = \frac{p_1}{\gamma+1}\mu^{\gamma\nu}\left[\frac{4\gamma}{c_1(\gamma+1)}\varphi^{\frac{1-\gamma}{2}} - (\gamma-1)\varphi^{-\gamma}\right]$$

$$\rho = \frac{2p_1}{\nu(\gamma^2-1)}\frac{\mu^{\nu-1}}{r}\frac{d}{d\xi}\left[\frac{4\gamma}{c_1(\gamma+1)}\varphi^{\frac{1-\gamma}{2}} - (\gamma-1)\varphi^{-\gamma}\right]$$

Thereby, $\phi(\xi) \geqslant 0$ is found from the equation

$$\left(\frac{\xi}{r^0}\right)^\nu + \frac{(\gamma^2-1)\nu}{2\sigma_\nu\gamma}\varphi - \frac{2}{c_1(\gamma+1)}\left(\frac{\xi}{r^0}\right)^\nu\varphi^{\frac{\gamma+1}{2}} = 0$$

According to (16), the pressure change directly behind the shock wave front is given by the formula

$$p_2 = p_1\left[1 + \frac{\nu(\gamma-1)}{\sigma_\nu}R_2{}^{-\nu}\right] \qquad \left(R_2 = \frac{r_2}{r^0}\right)$$

In the particular case when $c_1 = 0$, $p_1 = 0$, we obtain the known solution [1] of the self-similar problem of the point-blast, for which the initial gas density varies in accordance with the law $\rho_1 = A_1 r^{-\omega}$, where A_1 is some constant.

It should be pointed out further, that the solutions studied here may be used for problems of motion of a gas in a plane, cylindrical or spherical piston. From the condition of equality of piston velocity and the velocity of gas particles adjacent to the piston, we have

$$\frac{1}{r_n}\frac{dr_n}{dt} = -\frac{1}{\mu}\frac{d\mu}{dt}$$

where r_n is the radius of the piston.

From this we obtain $r_n = k_1/\mu$, where k_1 is a constant of integration. Using (1), we find the piston velocity

$$\frac{dr_n}{dt} = \mp k_1(A + B\mu^{\nu(\gamma-1)})^{\frac{1}{2}}$$

If $\mu(t)$ is known, and the arbitrary function $P(x)$ is also found, then the piston problem is solved.

The authors express their deep gratitude to L.I. Sedov for his interest in this work and his valuable advice.

BIBLIOGRAPHY

1. Sedov, L.I., *Metody podobiia i rasmernosti v mekhanike* (*Methods of similarity and dimensional analysis in mechanics*). 4th edition, Moscow, 1957.
2. Sedov, L.I., Ob integrirovanii uravnenii odnomernogo dvizheniia gaza (On the integration of the equations of one-dimensional gas flow). *Dokl. Akad. Nauk SSSR* Vol. 90, No. 5, 1953.
3. Keller, J.B., Spherical, cylindrical and one-dimensional gas flows. *Quart. Appl. Math.* Vol. 14, No. 2, pp. 171-184, 1956.

Translated by G.H.